

MULTIVALUED FUNCTIONALS, ONE-FORMS AND DEFORMED DE RHAM COMPLEX

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ABSTRACT. We discuss some applications of the Morse-Novikov theory to some problems in modern physics, where appears a non-exact closed 1-form ω (multi-valued functional). We focus mainly our attention to the cohomology $H_{\lambda\omega}^*(M^n, \mathbb{R})$ of the de Rham complex $\Lambda^*(M^n)$ of a compact manifold M^n with a deformed differential $d_\omega = d + \lambda\omega$. Using Witten's approach to the Morse theory one can estimate the number of critical points of ω in terms of $H_{\lambda\omega}^*(M^n, \mathbb{R})$ with sufficiently large values of λ (torsion-free Novikov's inequalities).

We show that for an interesting class of solvmanifolds G/Γ the cohomology $H_{\lambda\omega}^*(G/\Gamma, \mathbb{R})$ can be computed as the cohomology $H_{\lambda\omega}^*(\mathfrak{g})$ of the corresponding Lie algebra \mathfrak{g} associated with the one-dimensional representation $\rho_{\lambda\omega}$. Moreover $H_{\lambda\omega}^*(G/\Gamma, \mathbb{R})$ is almost always trivial except a finite number of classes $[\lambda\omega]$ in $H^1(G/\Gamma, \mathbb{R})$.

INTRODUCTION

In the beginning of the 80-th S.P. Novikov constructed ([N1], [N2]) an analogue of the Morse theory for smooth multi-valued functions, i.e. smooth closed 1-forms on a compact smooth manifold M . In particular he introduced the Morse-type inequalities (Novikov's inequalities) for numbers $m_p(\omega)$ of zeros of index p of a Morse 1-form ω .

In [N3], [Pa] a method of obtaining the torsion-free Novikov inequalities in terms of the de Rham complex of manifold was proposed. This method was based on Witten's approach [W] to the Morse theory. A. Pazhitnov generalized Witten's deformation $d + tdf$ (f is a Morse function on M) of the standard differential d in $\Lambda^*(M)$ by replacing df by an arbitrary Morse 1-form on M . For sufficiently large real values t one have the following estimate (torsion-free Novikov inequality [Pa]):

$$m_p(\omega) \geq \dim H_{t\omega}^p(M, \mathbb{R}),$$

where by $H_{t\omega}^p(M, \mathbb{R})$ we denote the p -th cohomology group of the de Rham complex $(\Lambda^*(M), d + t\omega)$ with respect to the new deformed differential $d + t\omega$.

Taking a complex parameter λ one can identify $H_{\lambda\omega}^*(M, \mathbb{C})$ with the cohomology $H_{\rho_{\lambda\omega}}^*(M_n, \mathbb{C})$ with coefficients in the local system $\rho_{\lambda\omega}$ of groups \mathbb{C} , where

$$\rho_{\lambda\omega}(\gamma) = \exp \int_{\gamma} \lambda\omega, \gamma \in \pi_1(M).$$

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L. Alania in [Al] studied $H_{\rho_{\lambda\omega}}^*(M_n, \mathbb{C})$ of a class of nilmanifolds M_n . He proved that $H_{\rho_{\lambda\omega}}^*(M_n, \mathbb{C})$ is trivial if $\lambda\omega \neq 0$. The proof was based on the Nomizu theorem [Nz] that reduce the problem to the computations in terms of the corresponding nilpotent Lie algebra. It was remarked in [Mil] that triviality of $H_{\rho_{\lambda\omega}}^*(G/\Gamma, \mathbb{R})$, with $\lambda\omega \neq 0$ follows from Dixmier's theorem [D], namely: *for a nilmanifold G/Γ the cohomology $H_\omega^*(G/\Gamma, \mathbb{R})$ coincides with the cohomology $H_\omega^*(\mathfrak{g})$ associated with the one-dimensional representation of the Lie algebra $\rho_\omega : \mathfrak{g} \rightarrow \mathbb{R}, \rho_\omega(\xi) = \omega(\xi)$ and hence $H_\omega^*(G/\Gamma, \mathbb{R}) = H_\omega^*(\mathfrak{g}) = 0$.*

Applying Hattori's theorem [H] one can observe that the isomorphism

$$H_\omega^*(G/\Gamma, \mathbb{R}) \cong H_\omega^*(\mathfrak{g})$$

still holds on for compact solvmanifolds G/Γ with completely solvable Lie group G . The calculations show that the cohomology $H_\omega^*(G/\Gamma, \mathbb{R})$ can be non-trivial for certain values $[\omega] \in H^1(G/\Gamma, \mathbb{R})$. However there exist only a finite number of such values.

Let us consider a finite subset $\Omega_{G/\Gamma}$ in $H^1(G/\Gamma, \mathbb{R}) \cong H^1(\mathfrak{g})$:

$$\Omega_{G/\Gamma} = \{\alpha_{i_1} + \dots + \alpha_{i_s} \mid 1 \leq i_1 < \dots < i_s \leq n, \ s = 1, \dots, n\},$$

where the set $\{\alpha_1, \dots, \alpha_n\}$ of closed 1-forms is in fact the set of the weights of completely reducible representation associated to the adjoint representation of \mathfrak{g} . It was proved in [Mil]: *if $-\omega \notin \Omega_{G/\Gamma}$, then the cohomology $H_\omega^*(G/\Gamma, \mathbb{R})$ is trivial.*

1. DIRAC MONOPOLE, MULTIVALUED ACTIONS AND FEINMAN QUANTUM AMPLITUDE

The notion of multivalued functional originates from topological study of the quantization process of the motion of a charged particle in the field of a Dirac monopole [Dir]. The Kirchhoff-Thomson equations for free motion of solids in a perfect noncompressible liquid also can be reduced to the theory of a charged particle on the sphere S^2 with some metric $g_{\alpha\beta}$ in a potential field U and in an effective magnetic field $F = F_{12}$ with a non-zero flux $4\pi s$ through S^2 . Locally (in some domain U_α) on the sphere we have the following formula for the action $S_\alpha(\gamma)$:

$$(1) \quad S_\alpha(\gamma) = \int_\gamma \left(\frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - U + e A_k^\alpha \dot{x}^k \right) dt,$$

where

$$(2) \quad x^1 = \theta, x^2 = \varphi, \quad F_{12} d\theta \wedge d\varphi = d(A_k^\alpha dx^k), \quad \int \int_{S^2} F_{12} d\theta \wedge d\varphi = 4\pi s \neq 0.$$

One can consider Feynman's paths integral approach to the quantization of the problem considered above. Recall that in the standard situation of single-valued action S , we consider the amplitude

$$\exp \{2\pi i S(\gamma)\}, \quad \gamma \in \Omega(x, x')$$

and the propagator

$$K(x, x') = \int_{\Omega(x, x')} \exp \{2\pi i S(\gamma)\} D\gamma.$$

For the Dirac monopole one can consider the set $\{S_1, S_2\}$ of local actions where $U_1 = S^2 \setminus P_N$ and $U_2 = S^2 \setminus P_S$, by P_N, P_S we denote the poles of the sphere S^2 .

Taking the equator γ with the positive orientation, one can easily test the ambiguity of the action:

$$(3) \quad S_1(\gamma) - S_2(\gamma) = e \int_{\gamma} (A_k^1 dx^k - A_k^2 dx^k) = e \int \int_{S^2} F_{12} d\theta \wedge d\varphi = 4\pi s e \neq 0.$$

The monopole is quantized if and only if the amplitude $\exp \{2\pi i S_{\alpha}(\gamma)\}$ is a single-valued functional, i.e. for an arbitrary closed $\gamma \in U_1 \cap U_2$ we have

$$\exp \{2\pi i S_1(\gamma)\} = \exp \{2\pi i S_2(\gamma)\}$$

The last condition is equivalent to the following one:

$$(4) \quad 4\pi s e = k, \quad k \in \mathbb{Z}.$$

Generalizing the situation with the Dirac monopole Novikov [N2] considered a n -dimensional manifold $M^n, n > 1$ with a metric g_{ij} , with a scalar potential U and with a two-form F of magnetic field not necessarily exact. In these settings one can consider a set of open $U_{\alpha} \subset M^n$, such that $F = F_{ij} dx^i \wedge dx^j$ is exact on U_{α} and $M^n \subset \cup_{\alpha} U_{\alpha}$. A 1-form $\omega_{\alpha} = A_k^{\alpha} dx^k$, $d\omega_{\alpha} = F_{ij} dx^i \wedge dx^j$ is determined up to some closed 1-form and we can consider the set of local actions:

$$(5) \quad S_{\alpha}(\gamma) = \int_{\gamma} \left(\frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - U + e A_k^{\alpha} \dot{x}^k \right) dt,$$

Let us consider a path $\gamma \subset U_{\alpha} \cap U_{\beta}$. The values $S_{\alpha}(\gamma)$ and $S_{\beta}(\gamma)$ do not coincide generally speaking. Hence the set $\{S_{\alpha}\}$ of local actions defines a multi-valued functional S . As $\omega_{\alpha} - \omega_{\beta}$ is closed on $U_{\alpha} \cap U_{\beta}$ the integral

$$S_{\alpha}(\gamma_{\lambda}) - S_{\beta}(\gamma_{\lambda}) = \int_{\gamma_{\lambda}} (\omega_{\alpha} - \omega_{\beta})$$

is invariant under any deformation $\gamma_{\lambda} \subset U_{\alpha} \cap U_{\beta}$ of γ in the class: a) periodic curves; b) the curves with the same end-points.

The crucial Novikov's observation was: the infinite-dimensional 1-form δS is well-defined and closed for the following function spaces:

- a) Ω^+ of the oriented closed contours γ , such that $\exists \alpha, \gamma \subset U_{\alpha}$;
- b) $\Omega(x, x')$ of the paths $\gamma(x, x')$ joining points x, x' , such that $\exists \alpha, \gamma(x, x') \subset U_{\alpha}$.

The set $\{S_{\alpha}\}$ of local actions determines also a multivalued (in general) functional $\exp \{2\pi i S\}$ on $\Omega(x, x')$. The local variational system $\{S_{\alpha}\}$ is quantized if and only if the Feinman quantum amplitude $\exp \{2\pi i S\}$ is a single-valued functional on Ω^+ . Or, in other words, for all $\gamma \in U_{\alpha} \cap U_{\beta}$ we have

$$(6) \quad \int_{\gamma} (\omega_{\alpha} - \omega_{\beta}) = k, \quad k \in \mathbb{Z}$$

If U_{α} and U_{β} are simply connected domains in M^n it is possible to consider a map $f : S^2 \rightarrow M$ such that γ is the image of the equator of the sphere S^2 and the images of two half-spheres of S^2 lie in U_{α} and U_{β} respectively. Then the condition (6) can be rewrited as

$$(7) \quad \int_{f(S^2)} F_{ij} dx^i \wedge dx^j = k, \quad k \in \mathbb{Z}.$$

Hence we can propose the following sufficient condition of the quantization: *a local variational system $\{S_{\alpha}\}$ on M^n that corresponds to some magnetic field $F = F_{ij} dx^i \wedge dx^j$ is quantized if F has integer-valued fluxes through all basic cycles of $H_2(M^n, \mathbb{Z})$.*

One can remark that the last condition is in fact excessive: it is sufficient to require integer-valued integrals of F over spheric cycles that lie in the image of the Hurevich map $H : \pi_2(M^n) \rightarrow H_2(M^n, \mathbb{Z})$.

2. AHARONOV-BOHM FIELD AND EQUIVALENT QUANTUM SYSTEMS

Another interesting example comes from Aharonov-Bohm experiment. We consider the electron move outside the ideal endless solenoid, i.e. the configuration space is $M = (\mathbb{R}^2 \setminus D_\varepsilon) \times \mathbb{R} = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 > \varepsilon^2\}$, where D_ε is two-dimensional disk of radius $\varepsilon \rightarrow 0$. The magnetic field $F = F_{ij} dx^i \wedge dx^j$ vanish outside solenoid, i.e. $F \equiv 0$ on M , hence

$$(8) \quad S_{\omega_\alpha}(\gamma) = \int_\gamma \frac{m\dot{x}^2}{2} dt + \omega_\alpha,$$

where $\omega_\alpha = eA_k dx^k$ is an arbitrary closed 1-form on M . The cohomology space

$$H^1(M, \mathbb{R}) = H^1(\mathbb{R}^2 \setminus D_\varepsilon, \mathbb{R}) = H^1(S^1, \mathbb{R}) = \mathbb{R}$$

is one-dimensional and hence

$$\omega_\alpha = \frac{e\Phi_\alpha}{2\pi} \frac{xdy - ydx}{x^2 + y^2} + df_\alpha,$$

for some constant Φ_α and function f_α on M .

Taking the circle $\gamma_0 = \partial D_\varepsilon = \{(\varepsilon \cos \varphi, \varepsilon \sin \varphi, 0), 0 \leq \varphi < 2\pi\}$ we have

$$\int_{\gamma_0} A_k^\alpha dx^k = \frac{1}{e} \int_{\gamma_0} \omega_\alpha = \Phi_\alpha = \int_{D_\varepsilon} F_{12} dx \wedge dy.$$

Hence the constant Φ_α is equal to the flux of the magnetic field F through the orthogonal section D_ε of our solenoid.

The form ω_α determines a representation ρ_{ω_α} of the fundamental group of M :

$$\rho_{\omega_\alpha} : \pi_1(M) \rightarrow \mathbb{C}^*, \quad \rho_{\omega_\alpha}(\gamma) = \exp \left\{ 2\pi i \int_\gamma \omega_\alpha \right\}, \quad \gamma \in \pi_1(M).$$

Let S_{ω_1} and S_{ω_2} be two actions for our system. They are quantummechanically equivalent if and only if

$$\exp \{ 2\pi i S_{\omega_1}(\gamma) \} = c(x, x') \exp \{ 2\pi i S_{\omega_2}(\gamma) \},$$

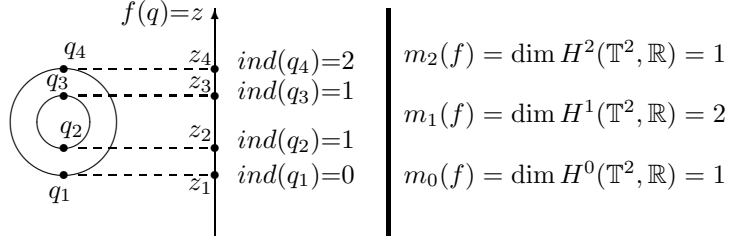
with a phase factor $c(x, x')$ depending only on end points x, x' of γ and $|c(x, x')| = 1$, i.e. $c(x, x')$ is physically unobservable. It is easy to show that the actions S_{ω_1} and S_{ω_2} are quantummechanically equivalent if and only if for any loop $\gamma \in \pi_1(M)$ the value of the integral $\int_\gamma (\omega_1 - \omega_2)$ is integer or, in other words, the form $(\omega_1 - \omega_2)$ has integer-valued integrals over basic cycles of $H_1(M, \mathbb{Z})$.

In our case $H_1(M, \mathbb{Z}) = \mathbb{Z}$ and the last condition is equivalent to the following one

$$(9) \quad \int_{\gamma_0} (\omega_1 - \omega_2) = e(\Phi_1 - \Phi_2) = k, \quad k \in \mathbb{Z}.$$

Hence (one of the important observations in the Aharonov-Bohm experiment) the fields with fluxes Φ_1 and Φ_2 , such that $\Phi_1 - \Phi_2 = \frac{k}{e}, k \in \mathbb{Z}$ can not be distinguished by any interference effect.

Now let us consider the case when M^n is not simply connected and the two-form F is globally exact on M^n (like in the Aharonov-Bohm experiment). Two solutions ω_1, ω_2 of the equation $d\omega = F_{ij} dx^i \wedge dx^j$ that correspond to two different actions

FIGURE 1. A height-function $f(q)=z$ for \mathbb{T}^2 .

$S_1(\gamma)$ and $S_2(\gamma)$ are determined up to a differential df by their integrals $\int_{\gamma_k} \omega_i$ over the basic cycles γ_k of $H_1(M^n, \mathbb{Z})$. These integrals can be interpreted as the fluxes of the continuation of F (with possible singularities) to some large manifold \tilde{M}^n . Two variational systems $S_1(\gamma)$ and $S_2(\gamma)$ are quantummechanically equivalent if and only if all integrals $\int_{\gamma_k} (\omega_1 - \omega_2)$ over basic cycles γ_k of $H_1(M^n, \mathbb{Z})$ are integer-valued.

The form $\omega_{12} = \omega_1 - \omega_2$ is a closed 1-form on M^n and it determines a representation $\rho_{\omega_{12}}$ of the fundamental group $\pi_1(M^n)$:

$$\rho_{\omega_{12}} : \pi_1(M^n) \rightarrow \mathbb{C}^*, \quad \rho_{\omega_{12}}(\gamma) = \exp \left\{ 2\pi i \int_{\gamma} \omega_{12} \right\}, \quad \gamma \in \pi_1(M^n).$$

Let M be a finite-dimensional (or infinite-dimensional) manifold and $S : M \rightarrow \mathbb{R}$ a function (functional) on it.

What are the relations between the set of the stationary points $dS = 0$ ($\delta S = 0$) and the topology of the manifold M ?

If S is a Morse function (generic situation), i.e. d^2S is non-degenerate at critical points, then one can define the Morse index $ind(P)$ of a critical point P of S as the number of negative squares of the quadratic form $d^2S(P)$ (if it is finite in the infinite-dimensional case).

Under some natural hypotheses the following inequality can be established:

$$m_p(S) \geq b_p(M) = \dim H^p(M).$$

3. SEMICLASSICAL MOTION OF ELECTRON AND CRITICAL POINTS OF 1-FORM

The semiclassical model of electron motion is an important tool for investigating conductivity in crystals under the action of a magnetic field. In the same time it is one of the most important examples of applications of topological methods in the modern physics.

Let us consider the corresponding quantum system defined for some crystal lattice $L = \mathbb{Z}^3$. Its eigenstates are the Bloch functions ψ_p . The particle quasimomentum p is defined up to a vector of the dual lattice $L^* = \mathbb{Z}^3$. Hence one can regard the space of quasimomenta as 3-dimensional torus $\mathbb{T}^3 = \mathbb{R}^3 / \mathbb{Z}^3$. The state energy $\varepsilon(p)$ is thus a function on \mathbb{T}^3 , i.e. 3-periodical function in \mathbb{R}^3 .

An external homogeneous constant magnetic field is a constant vector $H = (H_1, H_2, H_3)$ or in other words it is a 1-form $\omega = H_1 dp_1 + H_2 dp_2 + H_3 dp_3$ with constant coefficients.

The semiclassical trajectories projected to the space of quasimomenta are connected components of the intersection of the planes $(p, H) = \text{const}$ with constant energy surfaces $\varepsilon(p) = \text{const}$.

The constant energy surfaces $\varepsilon(p) = \varepsilon_F$ that correspond to the Fermi energies ε_F are called the Fermi surfaces. There are nonclosed trajectories on the Fermi surfaces with asymptotic directions and this topological fact explains an essential anisotropy of the metal conductivity at low temperatures.

One can study the intersections

$$(p, H) = c_0, \quad \varepsilon(p) = \varepsilon_0$$

as the level surfaces of the 1-form

$$\omega_{\varepsilon_0} = (H_1 dp_1 + H_2 dp_2 + H_3 dp_3)|_{\hat{M}_{\varepsilon_0}},$$

where 2-dimensional manifold

$$\hat{M}_{\varepsilon_0} = \{p \in \mathbb{R}^3 | \varepsilon(p) = \varepsilon_0\}$$

is the universal covering of the compact Fermi surface $\varepsilon(p) = \varepsilon_0$ in \mathbb{T}^3 . The last one we will denote also by M_{ε_0} . We can treat the 3-periodic form ω_{ε_0} as a 1-form on the compact manifold M_{ε_0} (we will keep the same notation for it).

The information about critical points of ω_{ε_0} is very important in the problem considered above. A generic 1-form ω_{ε_0} is a Morse form and has finitely many critical points on M_{ε_0} .

4. WITTEN'S DEFORMATION OF DE RHAM COMPLEX AND MORSE-NOVIKOV THEORY

In 1982 E. Witten proposed a new beautiful proof of the Morse inequalities using some analogies with supersymmetry quantum mechanics. Taking an arbitrary smooth real-valued function f on a Riemannian manifold M^n he considered a new deformed differential d_t in the de Rham complex $\Lambda^*(M^n)$ (t is a real parameter):

$$(10) \quad \begin{aligned} d_t &= e^{-f} d e^{f t} = d + t df \wedge, \\ d_t(\xi) &= d\xi + t df \wedge \xi, \quad \xi \in \Lambda^*(M^n), \end{aligned}$$

where d is the standard differential in $\Lambda^*(M^n)$:

$$(11) \quad \begin{aligned} d &: \Lambda^p(M^n) \rightarrow \Lambda^{p+1}(M^n), \\ \xi &= \sum_{i_1 < \dots < i_p} \xi_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} \in \Lambda^p(M^n), \\ d\xi &= \sum_{i_1 < \dots < i_p} \sum_q \frac{\partial \xi_{i_1 \dots i_p}}{\partial x^q} dx^q \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \in \Lambda^{p+1}(M^n). \end{aligned}$$

Taking arbitrary smooth vector fields X_1, \dots, X_{p+1} on M^n we have also the following formula:

$$(12) \quad \begin{aligned} d\xi(X_1, \dots, X_{p+1}) &= \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \xi([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{p+1}) + \\ &\quad + \sum_i (-1)^{i+1} X_i \xi(X_1, \dots, \hat{X}_i, \dots, X_{p+1}). \end{aligned}$$

We recall that that a differential p -form ξ is called closed if $d\xi = 0$ and it is called exact if $\xi = d\xi'$ for some $(p-1)$ -form ξ' . As $d^2 = 0$ the space of exact forms is a subspace of the space of closed ones and the p -th de Rham cohomology group $H^p(M^n, \mathbb{R})$ of the manifold M^n is defined as a quotient space of closed p -forms

modulo exact ones. In the same manner the cohomology $H_t^*(M^n, \mathbb{R})$ of the de Rham complex with respect to the deformed differential d_t can be defined.

The operators d_t and d are conjugated by the invertible operator e^{ft} and therefore the cohomology groups $H^*(M^n, \mathbb{R})$ (the standard de Rham cohomology) and $H_t^*(M^n, \mathbb{R})$ (the new ones) are isomorphic to each other. On the level of the forms this isomorphism is given by the gauge transformation

$$\xi \rightarrow e^{ft}\xi.$$

One can define the adjoint operator $d_t^* = e^{ft}d^*e^{-ft}$ with respect to the scalar product of differential forms

$$(\alpha, \beta) = \int_{M^n} (\alpha, \beta)_x dV,$$

where $(\alpha, \beta)_x$ is a scalar product in the bundle $\Lambda^*(T_x^*(M^n))$ evaluated with respect to the Riemannian metric g_{ij} of M^n and dV is the corresponding volume form.

One can also consider the deformed Laplacian $H_t = d_t d_t^* + d_t^* d_t$ acting on forms. An arbitrary element ω from $H_t^p(M^n, \mathbb{R})$ can be uniquely represented as an eigenvector with zero eigenvalue of the Hamiltonian $H_t = d_t d_t^* + d_t^* d_t$. Hence one can compute the Betti number $b_p(M^n) = \dim H^p(M^n, \mathbb{R})$ as the number of zero eigenvalues of H_t acting on p -forms.

It can be calculated that

$$(13) \quad H_t = d_t d_t^* + d_t^* d_t = dd^* + d^*d + t^2(df)^2 + t \sum_{i,j} \nabla_{(i,j)}^2(f) [\tilde{a}^i, \tilde{a}^{j*}],$$

where $(df)^2 = (df, df)_x = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j}$ and

$$\tilde{a}^i(\xi) = dx^i \wedge \xi, \quad \nabla_{(i,j)}^2 = \nabla_i \nabla_j - \Gamma_{ij}^k \nabla_k.$$

As the "potential energy" $t^2(df)^2$ of the Hamiltonian H_t becomes very large for $t \rightarrow +\infty$ the eigenfunctions of H_t are concentrated near the critical points $df = 0$ and the low-lying eigenvalues of H_t can be calculated by expanding about the critical points. Taking the Morse coordinates x^i in some neighbourhood W of a critical point P

$$f(x) = \frac{1}{2} \sum \lambda_i (x^i)^2, \quad \lambda_1 = \dots = \lambda_q = -1, \lambda_{q+1} = \dots = \lambda_n = 1,$$

where q is the index of the critical point P , and introducing a Riemannian metric g_{ij} on M^n such that x^i are Euclidean coordinates for g_{ij} in W one can locally evaluate the Hamiltonian H_t :

$$(14) \quad H_t = \sum_i \left(-\frac{\partial^2}{\partial x^{i2}} + t^2 x^{i2} + t \lambda_i [\tilde{a}^i, \tilde{a}^{i*}] \right).$$

The operator

$$H_i = -\frac{\partial^2}{\partial x^{i2}} + t^2 x^{i2}$$

is the Hamiltonian of the simple harmonic oscillator and it has the following set of eigenvalues

$$t(1 + 2N_i), \quad N_i = 0, 1, 2, \dots$$

with simple multiplicities. The operator H_i commutes with $[\tilde{a}^i, \tilde{a}^{i*}]$ and the eigenvalues of the last operator are equal to ± 1 :

$$[\tilde{a}^i, \tilde{a}^{i*}](\psi(x)dx^{i_1} \wedge \dots \wedge dx^{i_p}) = \begin{cases} \psi(x)dx^{i_1} \wedge \dots \wedge dx^{i_p}, & i \in (i_1, \dots, i_p), \\ -\psi(x)dx^{i_1} \wedge \dots \wedge dx^{i_p}, & i \notin (i_1, \dots, i_p). \end{cases}$$

Hence the eigenvalues of the restriction $H_t|_W$ are equal to

$$(15) \quad t \sum_i (1 + 2N_i + \lambda_i l_i), \quad N_i = 0, 1, 2, \dots, \quad l_i = \pm 1.$$

The corresponding eigenfunctions $\Psi_t = \psi(x, t)dx^{i_1} \wedge \dots \wedge dx^{i_p}$ are defined in W and not on the whole manifold M^n . Using the partition of unit one can define a new smooth q -form $\tilde{\Psi}_t$ on M^n such that $\tilde{\Psi}_t$ coincides with Ψ_t in some $\tilde{W} \subset W$ and $\tilde{\Psi}_t \equiv 0$ outside of W . The q -form $\tilde{\Psi}_t$ is called a quasi-mode:

$$(16) \quad H_t \tilde{\Psi}_t = t \left(\sum_i (1 + 2N_i + \lambda_i l_i) + \frac{B}{t} + \frac{C}{t^2} + \dots \right) \tilde{\Psi}_t, \quad t \rightarrow +\infty.$$

The numbers $t \sum_i (1 + 2N_i + \lambda_i l_i)$ are called asymptotic eigenvalues and the minimal value E_0^{as} of them approximates the minimal eigenvalue of H_t as $t \rightarrow +\infty$.

In order to find E_0^{as} , we must set $N_i = 0$ for all i . The sum

$$\sum_{i=1}^q (1 - l_i) + \sum_{i=q+1}^n (1 + l_i).$$

is non-negative and it is equal to zero if and only if

$$l_1 = \dots = l_q = 1, \quad l_{q+1} = \dots = l_n = -1.$$

This means that, H_t has precisely one zero asymptotic eigenvalue for each critical point of index q . Hence we have precisely $m_q(f)$ asymptotic zero eigenvalues (for q -forms). Vanishing of the first term of the asymptotical expansion (16) for a minimal eigenvalue of H_t is only a necessary condition to have zero energy level, hence the number $b_q(M^n)$ of zero eigenvalues does not exceed the number of zero asymptotic eigenvalues. In other words we have established the Morse inequalities

$$m_q(f) \geq b_q(M^n).$$

It was A. Pajitnov who remarked that it is possible to apply Witten's approach to the Morse-Novikov theory [Pa]. Let ω be a closed 1-form on M^n and t a real parameter. As in the construction above one can define a new deformed differential $d_{t\omega}$ in $\Lambda^*(M)$

$$d_{t\omega} = d + t\omega \wedge, \quad d_{t\omega}(\xi) = d\xi + t\omega \wedge \xi.$$

If the 1-form ω is not exact, the cohomology $H_{t\omega}^*(M, \mathbb{R})$ of the de Rham complex with the deformed differential $d_{t\omega}$ generally speaking is not isomorphic to the standard one $H^*(M, \mathbb{R})$. But $H_{t\omega}^*(M, \mathbb{R})$ depends only on the cohomology class of ω : for any pair ω, ω' of 1-forms such that $\omega - \omega' = d\phi$, where ϕ is a smooth function on M^n the cohomology $H_{t\omega}^*(M, \mathbb{R})$ and $H_{t\omega'}^*(M, \mathbb{R})$ are isomorphic to each other. This isomorphism can be given by the gauge transformation

$$\xi \rightarrow e^{t\phi} \xi; \quad d \rightarrow e^{t\phi} d e^{-t\phi} = d + t d\phi \wedge.$$

It is convenient also to consider instead of t a complex parameter λ . It was remarked in [N3], [Pa] that the cohomology $H_{\lambda\omega}^*(M, \mathbb{C})$ of $\Lambda^*(M)$ with respect to the deformed

differential $d_{\lambda\omega}$ coincides with the cohomology $H_{\rho_{\lambda\omega}}^*(M, \mathbb{C})$ with coefficients in the representation $\rho_{\lambda\omega} : \pi_1(M) \rightarrow \mathbb{C}^*$ of fundamental group defined by the formula

$$\rho_{\lambda\omega}([\gamma]) = \exp \int_{\gamma} \lambda\omega, \quad [\gamma] \in \pi_1(M),$$

We denote corresponding Betti numbers by $b_p(\lambda, \omega)$, $b_p(\lambda, \omega) = \dim H_{\rho_{\lambda\omega}}^*(M, \mathbb{C})$.

There is another interpretation of $H_{\rho_{\lambda\omega}}^*(M, \mathbb{C})$: the representation $\rho_{\lambda\omega} : \pi_1(M) \rightarrow \mathbb{C}^*$ defines a local system of groups \mathbb{C}^* on the manifold M . The cohomology of M with coefficients in this local system coincides with $H_{\rho_{\lambda\omega}}^*(M, \mathbb{C})$.

Now we can assume that ω is a Morse 1-form, i.e., in a neighborhood of any point $\omega = df$, where f is a Morse function. In other words ω gives a multi-valued Morse function. The zeros of ω are isolated, and one can define the index of each zero. The number of zeros of ω of index p is denoted by $m_p(\omega)$.

Following Witten's scheme A. Pazhitnov showed in [Pa] that for sufficiently large real numbers λ

$$m_p(\omega) \geq b_p(\lambda, \omega).$$

5. SOLVMANIFOLDS AND LEFT-INVARIANT FORMS

A solvmanifold (nilmanifold) M is a compact homogeneous space of the form G/Γ , where G is a simply connected solvable (nilpotent) Lie group and Γ is a lattice in G .

Let us consider some examples of solvmanifolds (first two of them are nilmanifolds):

- 1) a n -dimensional torus $T^n = \mathbb{R}^n / \mathbb{Z}^n$;
- 2) the Heisenberg manifold $M_3 = \mathcal{H}_3 / \Gamma_3$, where \mathcal{H}_3 is the group of all matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R},$$

and a lattice Γ_3 is a subgroup of matrices with integer entries $x, y, z \in \mathbb{Z}$.

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the only one non-trivial structure relation: $[e_1, e_2] = e_3$. The left invariant 1-forms on \mathcal{H}_3

$$(17) \quad e^1 = dx, \quad e^2 = dy, \quad e^3 = dz - xdy$$

are dual to e_1, e_2, e_3 and

$$(18) \quad de^1 = 0, \quad de^2 = 0, \quad de^3 = d(dz - xdy) = -dx \wedge dy = -e^1 \wedge e^2.$$

Now we are going to consider examples of solvmanifolds that are not nilmanifolds.

- 3) let G_1 be a solvable Lie group of matrices

$$(19) \quad \begin{pmatrix} e^{kz} & 0 & 0 & x \\ 0 & e^{-kz} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $e^k + e^{-k} = n \in \mathbb{N}, k \neq 0$.

G_1 can be regarded as a semidirect product $G_1 = \mathbb{R} \ltimes \mathbb{R}^2$ where \mathbb{R} acts on \mathbb{R}^2 (with coordinates x, y) via

$$z \rightarrow \phi(z) = \begin{pmatrix} e^{kz} & 0 \\ 0 & e^{-kz} \end{pmatrix}.$$

A lattice Γ_1 in G_1 is generated by the following matrices:

$$\begin{pmatrix} e^k & 0 & 0 & 0 \\ 0 & e^{-k} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & u_1 \\ 0 & 1 & 0 & v_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & u_2 \\ 0 & 1 & 0 & v_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \neq 0$.

The corresponding Lie algebra \mathfrak{g}_1 has the following basis:

$$e_1 = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & -k & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and the following structure relations:

$$[e_1, e_2] = ke_2, \quad [e_1, e_3] = -ke_3, \quad [e_2, e_3] = 0.$$

The left-invariant 1-forms

$$(20) \quad e^1 = dz, \quad e^2 = e^{-kz} dx, \quad e^3 = e^{kz} dy$$

are the dual basis to e_1, e_2, e_3 and

$$(21) \quad de^1 = 0, \quad de^2 = -ke^{-kz} dz \wedge dx = -ke^1 \wedge e^2, \quad de^3 = ke^1 \wedge e^3.$$

As the solvable Lie group G is simply connected the fundamental group $\pi_1(G/\Gamma)$ is naturally isomorphic to the lattice Γ : $\pi_1(G/\Gamma) \cong \Gamma$.

The Lie algebra \mathfrak{g}_1 of G_1 considered above is an example of completely solvable Lie algebra. A Lie algebra \mathfrak{g} is called completely solvable if $\forall X \in \mathfrak{g}$ operator $ad(X)$ has only real eigenvalues.

Let G/Γ be a solvmanifold. One can identify its de Rham complex $\Lambda^*(G/\Gamma)$ with the subcomplex in $\Lambda^*(G)$

$$\Lambda_{\Gamma inv}^*(G) \subset \Lambda^*(G)$$

of left-invariant forms on G with respect to the action of the lattice Γ .

The subcomplex $\Lambda_{\Gamma inv}^*(G)$ contains in its turn the subcomplex $\Lambda_{\Gamma inv}^*(G)$ of left-invariant forms with respect to the action of G .

Taking left-invariant vector fields X_1, \dots, X_{p+1} and a left-invariant p -form $\xi \in \Lambda_{G inv}^*(G)$ in the formula (12) we have:

$$(22) \quad d\xi(X_1, \dots, X_{p+1}) = \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \xi([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{p+1}).$$

The Lie algebra of left-invariant vector fields on G is naturally isomorphic to the tangent Lie algebra \mathfrak{g} . Hence one can identify the space $\Lambda_{G inv}^p(G)$ with the space $\Lambda^p(\mathfrak{g}^*)$ of skew-symmetric polylinear functions on \mathfrak{g} .

The differential d defined by (22) provides us with the cochain complex of the Lie algebra \mathfrak{g} :

$$(23) \quad \mathbb{R} \xrightarrow{d_0=0} \mathfrak{g}^* \xrightarrow{d} \Lambda^2(\mathfrak{g}^*) \xrightarrow{d} \Lambda^3(\mathfrak{g}^*) \xrightarrow{d} \dots$$

The dual of the Lie bracket $[\cdot, \cdot] : \Lambda^2(\mathfrak{g}) \rightarrow \mathfrak{g}$ gives a linear mapping

$$\delta : \mathfrak{g}^* \rightarrow \Lambda^2(\mathfrak{g}^*).$$

Consider a basis e_1, \dots, e_n of \mathfrak{g} and its dual basis e^1, \dots, e^n . Then we have the following relation:

$$(24) \quad de^k = -\delta e^k = -\sum_{i < j} C_{ij}^k de^i \wedge de^j,$$

where $[e_i, e_j] = \sum C_{ij}^k e_k$. The differential d is completely determined by (24) and the following property:

$$d(\xi_1 \wedge \xi_2) = d\xi_1 \wedge \xi_2 + (-1)^{\deg \xi_1} \xi_1 \wedge d\xi_2, \quad \forall \xi_1, \xi_2 \in \Lambda^*(\mathfrak{g}^*).$$

Cohomology of the complex $(\Lambda^*(\mathfrak{g}^*), \delta)$ is called the cohomology (with trivial coefficients) of the Lie algebra \mathfrak{g} and is denoted by $H^*(\mathfrak{g})$.

Let us consider the inclusion

$$\psi : \Lambda^*(\mathfrak{g}) \rightarrow \Lambda^*(G/\Gamma).$$

Let G/Γ be a compact solvmanifold, where G is a completely solvable Lie group, then $\psi : \Lambda^*(\mathfrak{g}) \rightarrow \Lambda^*(G/\Gamma)$ induces the isomorphism $\psi^* : H^*(\mathfrak{g}) \rightarrow H^*(G/\Gamma, \mathbb{R})$ in cohomology (Hattori's theorem [H], Nomizu's theorem for nilmanifolds [Nz]).

Let us return to our examples:

- 1) the cohomology classes $H^*(\mathbb{T}^n, \mathbb{R})$ are represented by invariant forms

$$dx^{i_1} \wedge \dots \wedge dx^{i_q}, \quad 1 \leq i_1 < \dots < i_q \leq n, \quad q = 1, \dots, n;$$

- 2) $H^*(\mathcal{H}_3/\Gamma_3, \mathbb{R})$ is spanned by the cohomology classes of the following left-invariant forms:

$$dx, dy, dy \wedge dz, dx \wedge (dz - xdy), dx \wedge dy \wedge dz.$$

- 3) $H^*(G_1/\Gamma_1, \mathbb{R})$ is spanned by the cohomology classes of:

$$e^1 = dz, \quad e^2 \wedge e^3 = dx \wedge dy, \quad e^1 \wedge e^2 \wedge e^3 = dx \wedge dy \wedge dz.$$

6. DEFORMED DIFFERENTIAL AND LIE ALGEBRA COHOMOLOGY

From the definition of Lie algebra cohomology it follows that $H^1(\mathfrak{g})$ is the dual space to $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$.

- 1) $b^1(\mathfrak{g}) = \dim H^1(\mathfrak{g}) \geq 2$ for a nilpotent Lie algebra \mathfrak{g} (Dixmier's theorem [D]);
- 2) $b^1(\mathfrak{g}) \geq 1$ for a solvable Lie algebra \mathfrak{g} ;
- 3) $b^1(\mathfrak{g}) = 0$ for a semi-simple Lie algebra \mathfrak{g} .

Consider a Lie algebra \mathfrak{g} with a non-trivial $H^1(\mathfrak{g})$. Let $\omega \in \mathfrak{g}^*, d\omega = 0$. One can define

- 1) a new deformed differential d_ω in $\Lambda^*(\mathfrak{g}^*)$ by the formula

$$d_\omega(a) = da + \omega \wedge a.$$

- 2) a one-dimensional representation

$$\rho_\omega : \mathfrak{g} \rightarrow \mathbb{K}, \quad \rho_\omega(\xi) = \omega(\xi), \quad \xi \in \mathfrak{g}.$$

Now we recall the definition of Lie algebra cohomology associated with a representation. Let \mathfrak{g} be a Lie algebra and $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ its linear representation. We denote by $C^q(\mathfrak{g}, V)$ the space of q -linear alternating mappings of \mathfrak{g} into V . Then one can consider an algebraic complex:

$$V = C^0(\mathfrak{g}, V) \xrightarrow{d} C^1(\mathfrak{g}, V) \xrightarrow{d} C^2(\mathfrak{g}, V) \xrightarrow{d} C^3(\mathfrak{g}, V) \xrightarrow{d} \dots$$

where the differential d is defined by:

$$(25) \quad \begin{aligned} (df)(X_1, \dots, X_{q+1}) &= \sum_{i=1}^{q+1} (-1)^{i+1} \rho(X_i)(f(X_1, \dots, \hat{X}_i, \dots, X_{q+1})) + \\ &+ \sum_{1 \leq i < j \leq q+1} (-1)^{i+j-1} f([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{q+1}). \end{aligned}$$

The cohomology of the complex $(C^*(\mathfrak{g}, V), d)$ is called the cohomology of the Lie algebra \mathfrak{g} associated to the representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$.

Let \mathfrak{g} be a Lie algebra and $\omega \in \mathfrak{g}^*$ is a closed 1-form. Then the complex $(\Lambda^*(\mathfrak{g}^*), d_\omega)$ coincides with the cochain complex of the Lie algebra \mathfrak{g} associated with the one-dimensional representation $\rho_\omega : \mathfrak{g} \rightarrow \mathbb{K}$, where $\rho_\omega(\xi) = \omega(\xi), \xi \in \mathfrak{g}$.

The proof follows from the formula:

$$(\omega \wedge a)(X_1, \dots, X_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} \omega(X_i)(a(X_1, \dots, \hat{X}_i, \dots, X_{q+1})).$$

The cohomology $H_\omega^*(\mathfrak{g})$ coincides with the Lie algebra cohomology with trivial coefficients if $\omega = 0$. If $\omega \neq 0$ the deformed differential d_ω is not compatible with the exterior product \wedge in $\Lambda^*(\mathfrak{g})$

$$d_\omega(a \wedge b) = d(a \wedge b) + \omega \wedge a \wedge b \neq d_\omega(a) \wedge b + (-1)^{\deg a} a \wedge d_\omega(b)$$

and the cohomology $H_\omega^*(\mathfrak{g})$ has no natural multiplicative structure.

Let G/Γ be a compact solvmanifold, where G is a completely solvable Lie group and $\tilde{\omega}$ is a closed 1-form on G/Γ . From the previous sections it follows that the cohomology $H_\omega^*(G/\Gamma, \mathbb{C})$ is isomorphic to the Lie algebra cohomology $H_\omega^*(\mathfrak{g})$ where $\omega \in \mathfrak{g}^*$ is the left-invariant 1-form that represents the class $[\tilde{\omega}] \in H^1(G/\Gamma, \mathbb{R})$.

One can define by means of ω a one-dimensional representation $\rho_\omega : G \rightarrow \mathbb{C}^*$:

$$\rho_\omega(g) = \exp \int_{\gamma(e, g)} \omega,$$

where $\gamma(e, g)$ is a path connecting the identity e with $g \in G$ (let us recall that G is a simply connected). As ω is the left invariant 1-form then

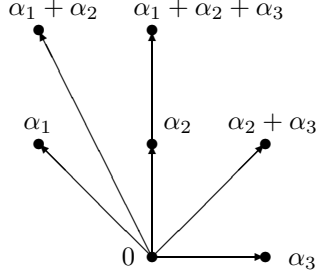
$$\int_{\gamma(e, g_1 g_2)} \omega = \int_{\gamma(e, g_1)} \omega + \int_{\gamma(g_1, g_1 g_2)} \omega = \int_{\gamma(e, g_1)} \omega + \int_{g_1^{-1} \gamma(e, g_2)} \omega$$

holds on and $\rho_\omega(g_1 g_2) = \rho_\omega(g_1) \rho_\omega(g_2)$.

The representation ρ_ω induces the representation of corresponding Lie algebra \mathfrak{g} (we denote it by the same symbol): $\rho_\omega(X) = \omega(X)$.

Let \mathfrak{g} be a n -dimensional real completely solvable Lie algebra (or complex solvable) and $b^1(\mathfrak{g}) = \dim H^1(\mathfrak{g}) = k \geq 1$. Then exists a basis e^1, \dots, e^n in \mathfrak{g}^* such that

$$(26) \quad \begin{aligned} de^1 &= \dots = de^k = 0, \\ de^{k+s} &= \alpha_{k+s} \wedge e^{k+s} + P_{k+s}(e^1, \dots, e^{k+s-1}), \quad s = 1, \dots, n-k. \end{aligned}$$

FIGURE 2. The finite subset $\Omega_{\mathfrak{g}} \subset H^1(\mathfrak{g})$.

where

$$(27) \quad \begin{aligned} \alpha_{k+s} &= \alpha_{s;1}e^1 + \alpha_{s;2}e^2 + \cdots + \alpha_{s;k}e^k, \\ P_{k+s}(e^1, \dots, e^{k+s-1}) &= \sum_{1 \leq i < j \leq k+s-1} P_{s;i,j} e^i \wedge e^j. \end{aligned}$$

It is convenient to define $\alpha_i = 0, i = 1, \dots, k$. The set $\{\alpha_1, \dots, \alpha_n\}$ of closed 1-forms is in fact the set of the weights of completely reducible representation associated to the adjoint representation $X \rightarrow ad(X)$.

For the proof we apply Lie's theorem to the adjoint representation ad restricted to the commutant $[\mathfrak{g}, \mathfrak{g}]$:

$$X \in \mathfrak{g} \rightarrow ad(X) : [\mathfrak{g}, \mathfrak{g}] \rightarrow [\mathfrak{g}, \mathfrak{g}].$$

Namely we can choose a basis e_{k+1}, \dots, e_n in $[\mathfrak{g}, \mathfrak{g}]$ such that the subspaces $V_i, i = k+1, \dots, n$ spanned by e_i, \dots, e_n are invariant with respect to the representation ad . Then we add e_1, \dots, e_k in order to get a basis of the whole \mathfrak{g} . For the forms of the dual basis e^1, \dots, e^n in \mathfrak{g}^* we have formulas (26).

Let us consider a new canonical basis of \mathfrak{g}^* :

$$(28) \quad \begin{aligned} \tilde{e}^1 &= e^1, \dots, \tilde{e}^k = e^k, \\ \tilde{e}^{k+s} &= t^{2(s-1)} e^{k+s}, \quad s = 1, \dots, n-k. \end{aligned}$$

where $t > 0$ is a real parameter.

Then for the differential d_ω in the complex $\Lambda^*(\tilde{e}^1, \dots, \tilde{e}^n)$ we have:

$$d_\omega = d_0 + \omega \wedge + t d_1 + t^2 d_2 + \dots, \quad d_0 \tilde{e}^i = \alpha_i \wedge \tilde{e}^i.$$

In particular

$$(d_0 + \omega \wedge)(\tilde{e}^{i_1} \wedge \cdots \wedge \tilde{e}^{i_q}) = (\alpha_{i_1} + \cdots + \alpha_{i_q} + \omega) \wedge \tilde{e}^{i_1} \wedge \cdots \wedge \tilde{e}^{i_q}.$$

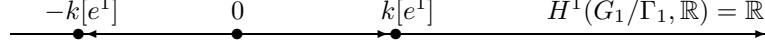
Now one can define the scalar product in $\Lambda^q(\tilde{e}^1, \dots, \tilde{e}^n)$ declaring the set $\{e^{i_1} \wedge \cdots \wedge e^{i_q}\}$ of basic q -forms as an orthonormal basis of $\Lambda^q(\tilde{e}^1, \dots, \tilde{e}^n)$. Then

$$(29) \quad \begin{aligned} d_\omega^* d_\omega + d_\omega d_\omega^* &= R_0 + t R_1 + t^2 R_2 + \dots, \\ R_0(\tilde{e}^{i_1} \wedge \cdots \wedge \tilde{e}^{i_q}) &= \|\alpha_{i_1} + \cdots + \alpha_{i_q} + \omega\|^2 \tilde{e}^{i_1} \wedge \cdots \wedge \tilde{e}^{i_q}. \end{aligned}$$

As $t \rightarrow 0$ the minimal eigenvalue of $d_\omega^* d_\omega + d_\omega d_\omega^*$ converges to the minimal eigenvalue of R_0 . Thus if

$$\alpha_{i_1} + \cdots + \alpha_{i_q} + \omega \neq 0, \quad 1 \leq i_1 < i_2 < \cdots < i_q \leq n$$

then $H_\omega^q(\mathfrak{g}) = 0$.

FIGURE 3. The finite subset Ω_{G_1/Γ_1}

Recall that $\alpha_1 = \dots = \alpha_k = 0$ and let us introduce the finite subset $\Omega_{\mathfrak{g}} \subset H^1(\mathfrak{g})$ such that:

$$(30) \quad \Omega_{\mathfrak{g}} = \{\alpha_{i_1} + \dots + \alpha_{i_s} \mid 1 \leq i_1 < \dots < i_s \leq n, \quad s = 1, \dots, n\}.$$

It follows that if

$$-\omega \notin \Omega_{\mathfrak{g}}$$

then the total cohomology $H_{\omega}^*(\mathfrak{g})$ is trivial: $H_{\omega}^*(\mathfrak{g}) \equiv 0$.

One can easily remark that the subset $\Omega_{\mathfrak{g}}$ is well-defined and does not depend on the ordering of weights α_i .

Let G/Γ be a compact solvmanifold, where G is a completely solvable Lie group. Then the left-invariant closed 1-forms from $\Omega_{\mathfrak{g}}$ define a finite subset in $H^1(G/\Gamma, \mathbb{R})$. We denote this subset by $\Omega_{G/\Gamma}$. Let ω be a closed 1-form on G/Γ . If the cohomology class

$$-[\omega] \notin \Omega_{G/\Gamma}$$

then the total cohomology $H_{\omega}^*(G/\Gamma, \mathbb{R})$ is trivial: $H_{\omega}^*(G/\Gamma, \mathbb{R}) \equiv 0$. The subset $\Omega_{G/\Gamma}$ is well-defined in terms of the corresponding Lie algebra \mathfrak{g} . The corresponding Lie algebra \mathfrak{g} must to be unimodular, i.e. the left-invariant n -form $e^1 \wedge \dots \wedge e^n$ determines non-exact volume form on G/Γ and hence

$$\alpha_1 + \alpha_2 + \dots + \alpha_n = 0.$$

If G/Γ is a compact nilmanifold then all the weights $\alpha_i, i = 1, \dots, n$ are trivial and therefore $\Omega_{G/\Gamma} = \{0\}$. Hence the cohomology $H_{\omega}^*(G/\Gamma, \mathbb{R})$ of a nilmanifold G/Γ is trivial if and only if the form ω is non-exact.

Let us consider a 3-dimensional solvmanifold G_1/Γ_1 defined in the previous section. We recall that the corresponding Lie algebra \mathfrak{g}_1 is defined by its basis e_1, e_2, e_3 and the following nontrivial brackets:

$$[e_1, e_2] = ke_2, \quad [e_1, e_3] = -ke_3.$$

For the dual basis of left-invariant 1-forms $e^1 = dz, e^2 = e^{-kz}dx, e^3 = e^{kz}dy$ we had

$$de^1 = 0, \quad de^2 = -ke^1 \wedge e^2, \quad de^3 = ke^1 \wedge e^3.$$

Hence $\alpha_1 = 0, \alpha_2 = -ke^1, \alpha_3 = ke^1$ and $\alpha_2 + \alpha_3 = 0$.

So it is easy to see that

$$\Omega_{G_1/\Gamma_1} = \{\pm k[e^1]\}$$

and therefore the cohomology $H_{\omega}^*(G_1/\Gamma_1, \mathbb{R})$ is trivial if $[\omega] \neq 0, \pm k[e^1]$.

a) $H_{k[e^1]}^*(G_1/\Gamma_1, \mathbb{R})$ is spanned by two classes:

$$e^2 = e^{-kz}dx, \quad e^1 \wedge e^2 = dz \wedge e^{-kz}dx.$$

b) $H_{-k[e^1]}^*(G_1/\Gamma_1, \mathbb{R})$ is spanned by two classes:

$$e^3 = e^{kz}dy, \quad e^1 \wedge e^3 = dz \wedge e^{kz}dy.$$

Hence we have the following Betti numbers $b_\omega^p = \dim H_\omega^p(G_1/\Gamma_1, \mathbb{R})$ of the solvmanifold G_1/Γ_1 :

$$(31) \quad \begin{aligned} 1) \quad & b_{\pm ke^1}^0 = 0, \quad b_{\pm ke^1}^1 = b_{\pm ke^1}^2 = 1, \quad b_{\pm ke^1}^3 = 0; \\ 2) \quad & b_0^0 = b_0^1 = b_0^2 = b_0^3 = 1. \end{aligned}$$

It was proved by G. Mostow in [Mos] that any compact solvmanifold G/Γ is a bundle with toroid as base space and nilmanifold as fibre, in particular a solvmanifold G/Γ is fibered over the circle $\pi : G/\Gamma \rightarrow S^1$. Hence the 1-form $\pi^*(d\varphi)$ on G/Γ has no critical points: $m_p(\pi^*(d\varphi)) = 0, \forall p$. It follows from A. Pajitnov's theorem [Pa] that for λ sufficiently large we have $H_{\lambda\pi^*(d\varphi)}^p(G/\Gamma, \mathbb{R}) = 0, \forall p$.

Now we are going to introduce an example of solvmanifold G/Γ with non completely solvable Lie group G . Let G_2 be a solvable Lie group of matrices

$$(32) \quad \begin{pmatrix} \cos 2\pi z & \sin 2\pi z & 0 & x \\ -\sin 2\pi z & \cos 2\pi z & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A lattice Γ_2 in G_2 is generated by the following matrices:

$$\begin{pmatrix} \cos \frac{2\pi n}{p} & \sin \frac{2\pi n}{p} & 0 & 0 \\ -\sin \frac{2\pi n}{p} & \cos \frac{2\pi n}{p} & 0 & 0 \\ 0 & 0 & 1 & \frac{n}{p} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & u_1 \\ 0 & 1 & 0 & v_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & u_2 \\ 0 & 1 & 0 & v_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where n is an integer, $p = 2, 3, 4, 6$ and $\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \neq 0$, or another type: $\tilde{\Gamma}_2$ is generated by the following matrices:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where n is an integer. The corresponding Lie algebra \mathfrak{g}_2 has the following basis:

$$e_1 = \begin{pmatrix} 0 & 2\pi & 0 & 0 \\ -2\pi & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and the following structure relations:

$$[e_1, e_2] = -2\pi e_3, \quad [e_1, e_3] = 2\pi e_2, \quad [e_2, e_3] = 0.$$

As the eigenvalues of $ad(e_1)$ are equal to $0, \pm 2\pi i$ the Lie group G_2 is not completely solvable.

The left-invariant 1-forms

$$(33) \quad e^1 = dz, \quad e^2 = \cos 2\pi z dx - \sin 2\pi z dy, \quad e^3 = \sin 2\pi z dx + \cos 2\pi z dy$$

are the dual basis to e_1, e_2, e_3 and

$$(34) \quad de^1 = 0, \quad de^2 = -2\pi e^1 \wedge e^3, \quad de^3 = 2\pi e^1 \wedge e^2.$$

The cohomology $H^*(\mathfrak{g}_2)$ is spanned by the cohomology classes of:

$$e^1, e^2 \wedge e^3, e^1 \wedge e^2 \wedge e^3.$$

But

$$\dim H^1(\mathfrak{g}_2) = 1 \neq \dim H^1(G_2/\Gamma_2, \mathbb{R}) = 3.$$

This example shows that, generally speaking, Hattori's theorem does not hold for non completely solvable Lie groups G , but the inclusion of left-invariant differential forms $\psi : \Lambda^*(\mathfrak{g}^*) \rightarrow \Lambda^*(G/\Gamma)$ always induces the injection ψ^* in cohomology.

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